Presentation of a System of Ordinary Differential Equations as an Equivalent Electrical Circuit

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Abstract – **In this paper the possibilities to present a system of ordinary differential equations describing any non-electrical object as an equivalent electrical circuit depending on the form of an ODE system are considered.**
Keywords – **ODE system, non-electrical object, electrical circuit, ALLTED.**

I. INTRODUCTION

During recent decades the usage of compound devices assumes increasing popularity in engineering and automobile industry, medicine, aerospace technology and requires appropriate designing tools. Taking into account that the electronic part is still a base for such devices, the most logical way to develop design tools is the transformation of the objects' mathematical models to the basis of the circuit design software. An additional argument for this is such compound devices are often produced using the same technology as integral circuits and could be their component parts.

A mathematical model of a physical object is usually presented as a system of non-linear algebraic-differential equations. However, the type of that system differs depending on the field, a mathematical model is described for, as well as the way of its obtaining. The typical examples are an analytical description of a mathematical model by Cauchy differential equations for classical mathematics, systems of implicitly defined non-linear algebraic-differential equations for electrical circuit description, partial derivative systems obtained by finite-element method for mechanics, hydraulics and so on. The transfer from one basis to other when synthesizing devices or subsystems is often hard to be formalized and automatized.

So, there is the actual task to present a non-electrical component of some system as an equivalent electrical circuit. This will allow making use of advantages of circuit design software to simulate such a system as a whole as opposed to the simulation of electrical and non-electrical components separately with a further accounting of the mutual influence of the system parts.

II. CAUCHY ODE SYSTEM

Let us consider how a certain non-electrical object described by the system of ordinary differential equations could be presented as an equivalent electrical circuit for further joint simulation with the electrical part of a compound system.

In a common case, the system of Cauchy ordinary differential equations could be presented as

$$y'_j(x) = f(x, y_i), i = 1(1)n, j = 1(1)n ,$$

where n is a number of the system's equations. Let us

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examine the features of building circuit equivalents of such a system for a number of separate cases.

1. Let ODE system be presented as

$$y'_j(x) = \sum_{i=1}^{n} f(y_i), j = 1(1)n \qquad (1)$$

and let's consider a proper example [1]:

$$\begin{cases} dy_1/dx = \mu_1 y_1 \\ dy_2/dx = y_1 + \mu_1 y_2 \\ dy_3/dx = \mu_2 y_3 \\ dy_4/dx = y_3 + \mu_2 y_4 \\ dy_5/dx = 2y_4 + \mu_2 y_5 \\ dy_6/dx = 3y_5 + \mu_2 y_6 \end{cases}$$

The exact solution of this linear system with a right-side matrix consisting of two Jordan cells is $(0 \le x \le 1)$:

$$\begin{cases} y_1(x) = y_1(0)e^{\mu_1 x} \\ y_2(x) = (y_2(0) + y_1(0)x)e^{\mu_1 x} \\ y_3(x) = y_3(0)e^{\mu_2 x} \\ y_4(x) = (y_4(0) + y_3(0)x)e^{\mu_2 x} \\ y_5(x) = (y_5(0) + 2y_4(0)x + y_3(0)x^2)e^{\mu_2 x} \\ y_6(x) = (y_6(0) + 3y_5(0)x + 3y_4(0)x^2 + y_3(0)x^3)e^{\mu_2 x} \end{cases}$$

Let parameters $\mu_1 = -1$, $\mu_2 = 2$, then the system looks like:

$$\begin{cases} dy_1/dx = -y_1 \\ dy_2/dx = y_1 - y_2 \\ dy_3/dx = 2y_3 \\ dy_4/dx = y_3 + 2y_4 \\ dy_5/dx = 2y_4 + 2y_5 \\ dy_6/dx = 3y_5 + 2y_6 \end{cases}$$

Under the initial conditions $y_1(0)=y_2(0)=2$, $y_3(0)=y_4(0)=y_5(0)=y_6(0)=10$ the exact solution is:

$$\begin{cases} y_1(x) = e^{-x} \\ y_2(x) = (1-x)e^{-x} \\ y_3(x) = 10e^{2x} \\ y_4(x) = (10+10x)e^{2x} \\ y_5(x) = (10+20x+10x^2)e^{2x} \\ y_6(x) = (10+30x+30x^2+10x^3)e^{2x} \end{cases}$$

The Matlab [2] corresponding function plots are presented on Figs. 1 and 2.
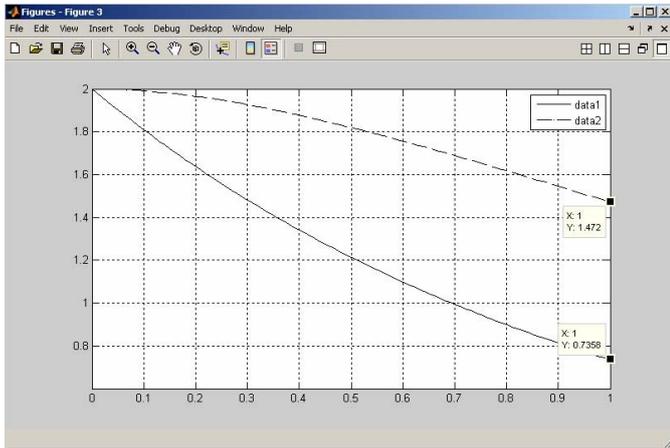
Fig.1 Matlab $y_1(x)$, $y_2(x)$ function plots.
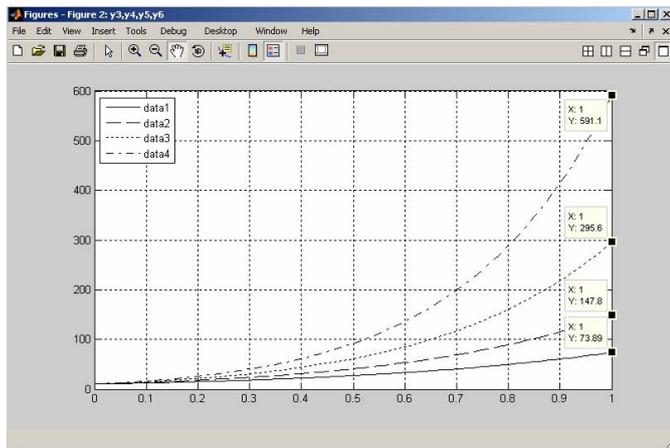
Fig.2 Matlab $y_3(x)$, $y_4(x)$, $y_5(x)$, $y_6(x)$ function plots.

Let us examine how that equation system can be presented as an appropriate equivalent electrical circuit.

Let us assume that the variable vector $y_1, y_2, ..., y_6$ is equal to the node voltages $V_1, V_2, ..., V_6$ of a certain equivalent circuit respectively. In that case, the first equation will look like:

$$\frac{dV_1}{dt} = -V_1, \text{ or } \frac{dV_1}{dt} + V_1 = 0. \qquad (2)$$

Taking into account that a C capacitance current can be presented as $i_C = C \frac{dU_C}{dt}$ while a G conductance current can be presented as $i_G = GU_G$, in concordance with Kirchhoff's first rule this equation could be matched to the circuit element equation:

$$C_1 \frac{dV_1}{dt} + G_{11}V_1 = 0. \qquad (3)$$

It is obvious that the equation (3) is identical to the equation (2) if C_1 is a unit capacitance while G_{11} is a unit conductance. Taking into account the signs, the respective circuit equivalent describing component equation (3) in the ALLTED [3] object description language looks like (Fig. 3):

```
C1(0,1)=1;
G11(0,1)=1;
```

For the second differential equation written down as

$$\frac{dV_2}{dx} - V_1 + V_2 = 0$$

the respective component equation looks like:

$$C_2 \frac{dV_2}{dt} + G_{21}V_1 + G_{22}V_2 = 0.$$

Since this component equation involves V_1 voltage, it is needed to add an element linearly depending on the first node's voltage in order to present a contribution of the respective current at the second node. Subject to the sign, the circuit equivalent for the second equation looks like (Fig. 4):

```
C2(0,2)=1;
G21(0,2)=FL(-1/UC1);
G22(0,2)=1;
```
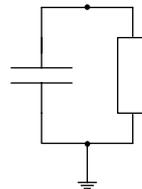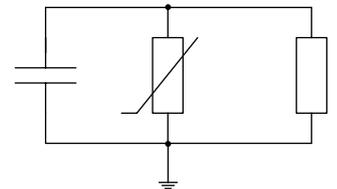

Fig.3 First circuit fragment.   Fig.4 Second circuit fragment.

After appropriate transformations made for all the six differential equations, the equivalent circuit for the entire equation system is obtained:

```
Object
Circuit DyfRiv1;
C1(0,1)=1;
G11(0,1)=1;
C2(0,2)=1;
G21(0,2)=FL(-1/UC1);
G22(0,2)=1;
C3(0,3)=1;
G31(0,3)=-2;
C4(0,4)=1;
G41(0,4)=FL(-1/UC3);
G42(0,4)=-2;
C5(0,5)=1;
G51(0,5)=FL(-2/UC4);
G52(0,5)=-2;
C6(0,6)=1;
G61(0,6)=FL(-3/UC5);
G62(0,6)=-2;
```

The proper task for simulation in the ALLTED task description language where the initial conditions are setting up by the INSTALL directive looks like:

```
&&
Task
TR;                     # Transient analysis
const nstep=50000;
const TMAX=1, minstep=1e-4;
INSTALL V1=2, V2=2;
INSTALL V3=10, V4=10, V5=10, V6=10;
plot V1,V2;
plot V3,V4,V5,V6;
&&
End
```

The ALLTED simulation results are presented on Figs. 5 and 6. So, it is obvious that the results of the differential equation system solution by means of the circuit design software coincide entirely with analytical ones.
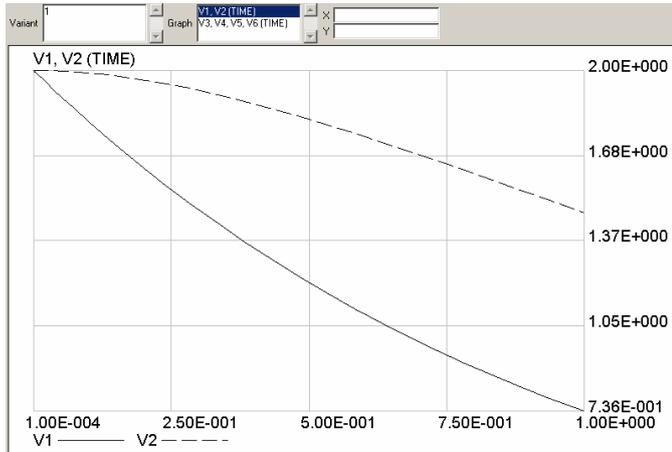

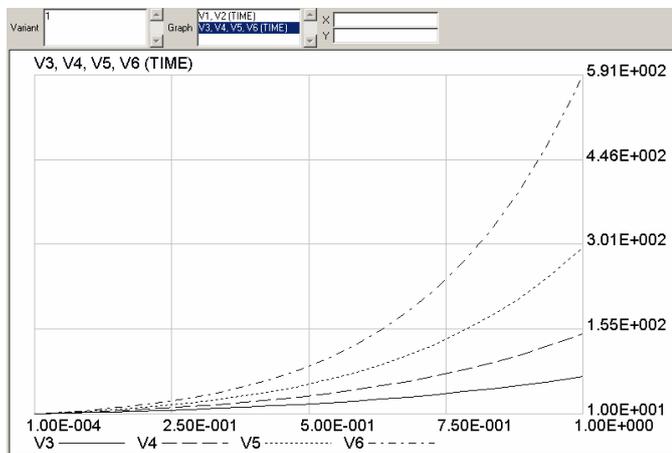Fig.5 ALLTED $y_1(x)$, $y_2(x)$ function plots.

Fig.6 ALLTED $y_3(x)$, $y_4(x)$, $y_5(x)$, $y_6(x)$ function plots.

2. Let ODE system be presented as

$$y'_j(x) = \sum_{i=1}^{n} f(y_i) + f(x), j = 1(1)n. \qquad (4)$$

and let's consider a proper example:

$$\begin{cases} dy_1/dx = 4y_1 - 3y_2 + \sin x \\ dy_2/dx = 2y_1 - y_2 - 2\cos x \end{cases}$$

Under the initial conditions $y_1(0)=1$, $y_2(0)=2$ the exact solution of this system is:

$$\begin{cases} y_1(x) = \cos x - 2\sin x \\ y_2(x) = 2\cos x - 2\sin x \end{cases}.$$

The plots of the respective functions build by Matlab software are presented on Fig. 7.

Since there is the explicit x dependence in the system in question, for the electrical circuit it is same as the presence of a current source with a value numerically equal to the respective time-dependent function.

In this case the equivalent circuit and the task for simulation looks like:

```
Object
Circuit DyfRiv2;
C1(0,1)=1;
G11(0,1)=-4;
G12(0,1)=FL(3/UC2);
J1(0,1)=FSIN(0,1,0.1592,0,0,0);
C2(0,2)=1;
G21(0,2)=FL(-2/UC1);
G22(0,2)=1;
J2(0,2)=FSIN(0,-2,0.1592,0,0,90);
&&
Task
TR;                      # Transient analysis
const TMAX=3.1415, minstep=1e-15;
INSTALL V1=1, V2=2;
plot V1,V2;
&&
End
```
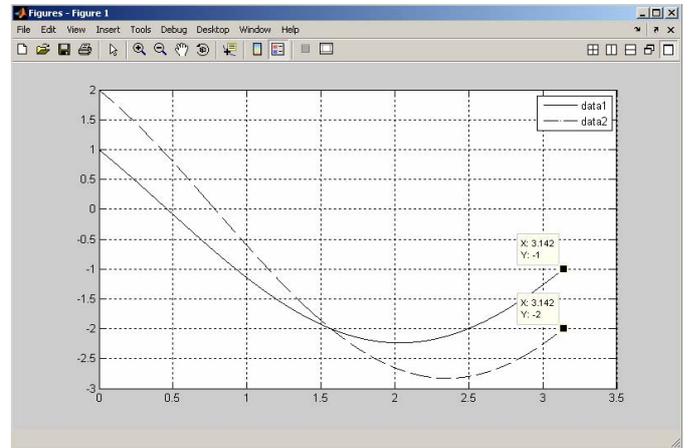

Fig.7 Matlab $y_1(x)$, $y_2(x)$ function plots.

It is evident that the current sources describing the equation system's respective functions are specified by

FSIN (*Ioff*, *Iampl*, *Freq*, *TD*, *DF*, *Phase*)

function where:
- *Ioff* is an initial offset value of the output current;
- *Iampl* is a peak amplitude of the output current;
- *Freq* is a frequency;
- *TD* is an initial signal delay;
- *DF* is a damping factor;
- *Phase* is a phase.

The third FSIN function's parameter is a frequency (its value is defined as a number of periods of function per time unit). Taking into account that the period of function is 2π sec in our case, the frequency in question is $1/2\pi$ Hz, i.e. 0.1592 approximately.

The last parameter of FSIN function is a phase whose value for the J2 source is specified as 90 that in fact transforms this function from sine to cosine.

The results of simulation by the ALLTED software are presented on Fig. 8. They indicate that the results of differential equation system solution by the circuit design software coincide fully with analytical ones.
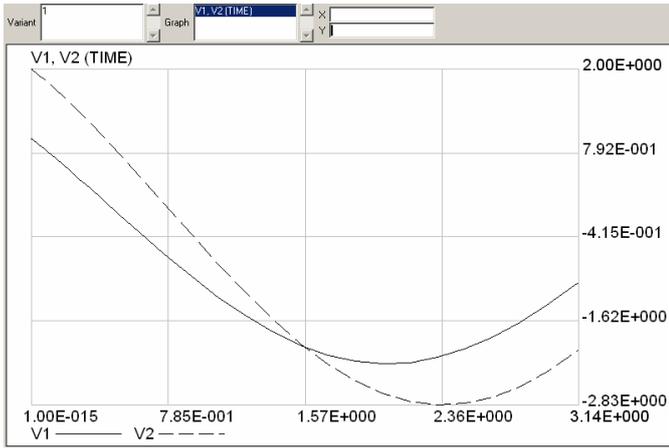
Fig.8 ALLTED $y_1(x)$, $y_2(x)$ function plots.

3. If the equation system could not be presented as (1) or (4), let's consider the possibilities to construct the equivalent circuit for the ODE system in a common case:

$$y'_j(x) = \sum_{i=1}^{n} f(x, y_i), \; j = 1(1)n$$

by an example:

$$\begin{cases} dy_1/dx = y_1^2 y_2 \\ dy_2/dx = -\dfrac{1}{y_1} \end{cases}.$$

Under the initial conditions $y_1(0)=y_2(0)=1$ the solution of this system is:

$$\begin{cases} y_1 = e^x \\ y_2 = e^{-x} \end{cases}.$$

The feature of this task is the one component of solution and its derivative increase rapidly but another one and its derivative decrease rapidly (Fig. 9).

Let's present the source equation system as:

$$\begin{cases} C_1 \dfrac{dV_1}{dt} + G_{11}V_1 = 0 \\ C_2 \dfrac{dV_2}{dt} + G_{21}V_2 = 0 \end{cases}.$$
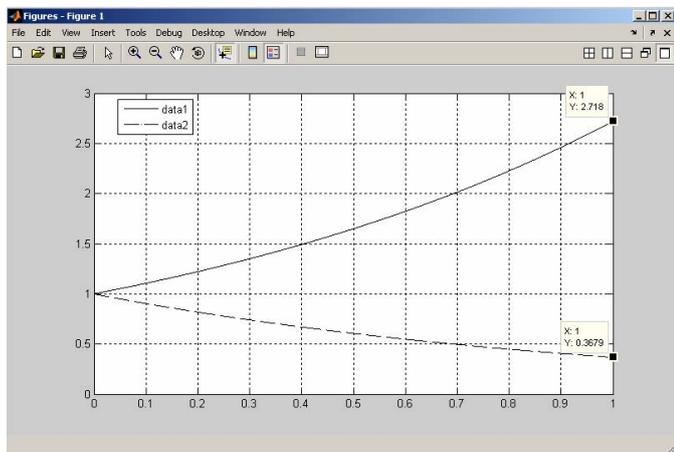

Fig.9 Matlab $y_1(x)$, $y_2(x)$ function plots.

Obviously that the conductance value for the first equation is $G_{11} = G_0 V_1 V_2$, and for the second one is $G_{21} = G_0 /(V_1 V_2)$ where G_0 is a respective coefficient and equals to 1 for the first equation and -1 for the second one.

There are two possibilities to integrate such systems in the ALLTED software: by providing non-linear functions via the input language translator, or by its addition into the function library. Let's consider both the possibilities.

Providing non-linear functions via the input language translator.

In this case the user-defined functions to define the respective current contribution have to be added to the circuit description.

```
Object
Circuit DyfRiv3;
C1(0,1)=1;
G1(0,1)=F1511(-1/UC1, UC2);
C2(0,2)=1;
G2(0,2)=F1512(1/UC1);
FUNCTION F1511(G0/X1,X2);
(G0*X1*X1)*X2;
FUNCTION F1512(G0/X1);
(G0/X1);
&&
Task
TR;                    # Transient analysis
const TMAX=1, minstep=1e-6;
INSTALL V1=1, V2=1;
plot V1, V2;
&&
End
```

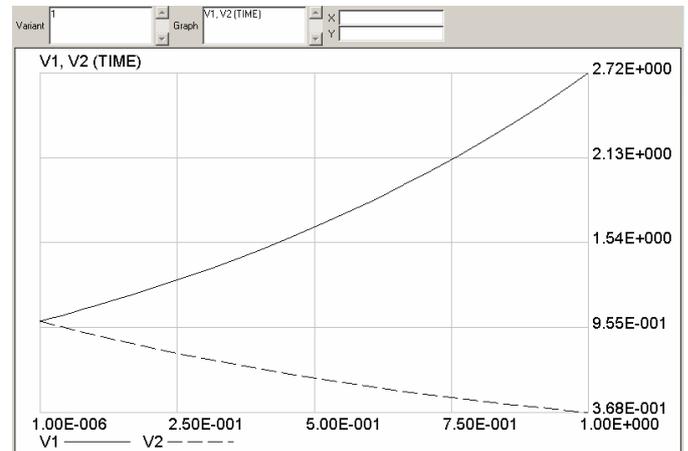The ALLTED simulation results are presented on Fig. 10.

Fig.10 ALLTED $y_1(x)$, $y_2(x)$ function plots.

Addition of non-linear functions into the function library.

In the case when the function shape depends on the analysis type, additional checks of the argument values are required etc. that could not be implemented by the input language it is required to add a non-linear user-defined function straight to the package body. For this purpose a user is provided with the access to the USERFX subroutine where the package's function set add-on takes place by the standard rules by the

high-level language. The disadvantage of this way is the necessity to recompile the USERFX subroutine that however does not affect the rest of the package subroutines.

The contribution of the $G_{11}V_1$ current to the first equation can be described by the F511(G_0/V1,V2) function that in the USERFX subroutine looks like:

```
511 Continue
G0=PAR(1)
V1=ARG(1)
V2=ARG(2)
YM=G0* V1*V1*V2
VAM(1)=2*G0*V1*V2
VAM(2)=G0*V1*V1
RETURN
```

where YM is a component current, VAM(1), VAM(2) are its partial derivatives.

Similarly, for the second equation the contribution of the $G_{21}V_2$ current is described by the F512(G_0/V1) function that looks like (TN is a variable keeping the current time value):

```
512 Continue
X=TN
G0=PAR(1)
V1=ARG(1)
IF(X.EQ.0) GO TO 5120
YM=G0/V1
VAM(1)=-G0/(V1*V1)
RETURN
5120 YM=1
VAM(1)=1
RETURN
```

The task file describing the solution of the system in question looks like:

```
Object
Circuit DyfRiv3;
C1(0,1)=1;
G1(0,1)=F511(-1/UC1, UC2);
C2(0,2)=1;
G2(0,2)=F512(1/UC1);
&&
Task
TR;                      # Transient analysis
const TMAX=1, minstep=1e-6;
INSTALL V1=1, V2=1;
plot V1, V2;
&&
End
```

The ALLTED simulation results are absolutely identical to that ones obtained by the previous way (Fig. 10).

The advantage of using one of these approaches is a possibility to implement a function of any shape but the complication to automatize the process of adding such a function can be mentioned as a disadvantage.

III. ODE System in a Matrix Form

Quite often different physical objects can be presented as a system of ordinary differential equations which could be written down in such a way:
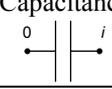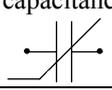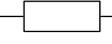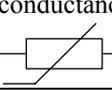
$$C\frac{dy}{dx} = Gy + B(x) , \qquad (5)$$

where y is a vector of flow-difference variables, x is an independent variable, C and G are coefficient matrices, B is an x-dependent function vector.

The presentation of an object description in such a form slightly differs from its presentation as Cauchy ordinary differential equations but in this case the above-mentioned ways to construct an equivalent electrical circuit could be used too.

The advantage of such a matrix presentation of the equation system which describes the object in question is a possibility to create a system to construct automatically an equivalent circuit of the mathematical model being considered.

The correspondence between matrix and electrical circuit elements is shown in Table 1 (*i=0(1)n*, *j=0(1)n*, *i≠j*, *n* is a matrix dimension).

TABLE 1
CORRESPONDENCE BETWEEN MATRIX AND
ELECTRICAL CIRCUIT ELEMENTS

Matrix element	Component	Component equation	Electrical circuit element
c_{ii}	Capacitance	$I_C = C\dfrac{dU_{i0}}{dt}$	C(0,i) = c_{ii}
c_{ij}	Non-linear capacitance	$I_C = C\dfrac{dU_{j0}}{dt}$	C(0,i) = FL(c_{ij}/U_{j0})
g_{ii}	Conductance	$I_G = GU_{i0}$	G(0,i) = g_{ii}
g_{ij}	Non-linear conductance	$I_G = GU_{j0}$	G(0,i) = FL(g_{ij}/U_{j0})
$b_i(x)$	Current source	$I_J = b_i(t)$	J(0,i) = $b_i(t)$

REFERENCES

[1] О.Б. Арушунян, С.Ф. Залеткин, Н.Н. Калиткин, "Тесты для вычислительного практикума по обыкновенным дифференциальным уравнениям", *Вычислительные методы и программирование*, т. 3, с. 11-19, 2002.

[2] MATLAB: http://www.mathworks.com/products/matlab/.

[3] A. Petrenko, V. Ladogubets, V. Tchkalov, Z. Pudlowski, *ALLTED – a computer-aided engineering system for electronic circuit design*, Melbourne: UICEE, 1997.

IV. CONCLUSION

It is shown that the approach stated allows getting an equivalent electrical circuit for an object model presented by a system of ordinary differential equations in a general form. If the system of ODEs can be reduced to the form (5), the equivalent electrical circuit could be constructed automatically due to the direct correspondence between the elements of matrices describing the system and the elements of the equivalent electrical circuit.