Sigma-Delta Modulators Transfer Functions Derivation

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***Abstract*—Sigma-delta modulator structure is presented in the form of matrix equations. The equations allow to easily obtain analytical expressions for the noise and signal transfer functions for arbitrary modulator structures. As a result the modulator structures analysis and comparison become straightforward.**

Keywords—ΣΔ modulator, signal flow graph, noise transfer function, signal transfer function, matrix equations.

I. INTRODUCTION

Today sigma-delta ADCs and DACs are used in a wide variety of digital signal processing applications – from high precision converters in measurement systems to high quality audio signal converters

The main part of such converters is sigma-delta modulator. A number of converter structures is proposed **[1, 2]**.

The most important characteristic of the modulators is noise transfer function which determines the achievable signal to noise ratio and consequently the effective converter word length.

Therefore one of the main tasks in sigma-delta modulator design is evaluation and investigation of the noise and signal transef functions and frequency responses.

Below a matrix model of the sigma-delta modulator is proposed. It allows to automatically obtain the transfer functions of the arbitrary structure modulators.

Two examples of transfer functions and frequency characteristics evaluation are presented for the modulators of arbitrary structure.

II. MATRIX MODEL

Sigma-delta modulator is a discrete system which in the first approximation may be regarded as linear and can be described in terms of difference equation. The exact form of this equation depends on realization structure. Sigma-delta modulators of high order are built with rather complicated structures using multiple feedback and feedforward connections [1, 2]. Due to this fact a problem of transfer function evaluation for a given structure arises.

A similar problem arises in analysis of digital fiters structures. For the solution of this problem a matrix model was developed. The model is based on the representation of a digital circuit of arbitrary topology as a directed flow graph.

A digital network of arbitrary structure is depicted as a digital circuit flow graph containing a number of branches and nodes.

We define two branch types: branches with delay and branches without delay. The branches without delays transfer signals from node to node with certain coefficient. The node output signal is the sum of signals of entering branches. So an arbitrary digital circuit can be described by a difference equation of the following form [3]:

$$\mathbf{w}(n) = \mathbf{C}x(n) + \mathbf{A}\mathbf{w}(n) + \mathbf{B}\mathbf{w}(n-1)$$
$$y(n) = \mathbf{D}\mathbf{w}(n)$$

where **w***(n)* – column vector (NNx1) of node signals, *x(n)* – scalar input signal, **A** – matrix NNxNN of coefficients without delays, B – matrix NNxNN of delays, **C** – column vector (NNx1) of inputs,-**D** – row vector (1xNN) of outputs, y(n) – output signal, NN – number of the signal graph nodes.

Flow graph nodes are numbered in an arbitrary order. In general case, when the equations are used for digital circuit simulation the matrix **A** must be transformed into low triangle form [3]. This transformation is done by renumbering of the graph nodes which results in the matrix **A** rows and columns rearrangement.

Taking z-transform of both sides of the difference equation system shown above and resolving the obtained equations against *Y(z)* we obtain transfer function of arbitrary digital circuit as

$$\mathbf{W}(z) = \mathbf{C}X(z) + \mathbf{A}\mathbf{W}(z) + z^{-1}\mathbf{B}\mathbf{W}(z)$$
$$Y(z) = \mathbf{D}\mathbf{W}(z)$$
$$H(Z) = \mathbf{D}(\mathbf{I} - \mathbf{A} - z^{-1}\mathbf{B})^{-1}\mathbf{C} = \mathbf{D}\mathbf{S}^{-1}\mathbf{C} \qquad (1)$$

where ***S*** – system matrix. For the purpose of transfer function calculation there is no need to transform the matrix **A** into low triangle form. The transformation does not influence the values of determinant of the matrix and the values of algebraic complements of the matrix elements..

III. EXAMPLES

Let us consider the sigma-delta modulator with feedforward topology presented in Fig. 2.9 of [2]. The corresponding flow graph is shawn in Fig. 1. In the figure flow graph nodes are shawn as circles with numbers inside. The nodes are numerated in an arbitrary order.

The input signal V_{in} is applied to the node 13. The output V_{out} is taken from the node 12. Quantization noise is applied to the node 12. The ribs that represent the forward links are not shawn in the figure. They transfer the output signals of the nodes 2, 5 and 8 to the node 12 with coefficients b1, b2 and b3 respectively. The singe feedback rib transfers the output of the node 12 to the node 13 with coefficient f1.

The size of the system matrix for the flow graph is 14x14. And only 35 matrix elements are nonzero. So to show here he full matrix is impractical, and we show only nonzero elements:

$$\mathbf{S}(\mathbf{i},\mathbf{i}) = 1, i = 1...14$$

$$\mathbf{S}(2,1) = \mathbf{S}(5,4) = \mathbf{S}(8,7) = \mathbf{S}(11,10) = \\ = \mathbf{S}(12,11) = -1$$

$$\mathbf{S}(1,2) = \mathbf{S}(1,14) = \mathbf{S}(4,3) = \mathbf{S}(4,5) = \\ = \mathbf{S}(7,6) = \mathbf{S}(7,8) = \mathbf{S}(10,9) = \mathbf{S}(10,11) = -z^{-1}$$

$$\mathbf{S}(14,13) = -a1 \quad \mathbf{S}(3,2) = -a2$$

$$\mathbf{S}(6,5) = -a3 \quad \mathbf{S}(9,8) = -a4$$

$$\mathbf{S}(13,12) = -f1$$

$$\mathbf{S}(12,2) = -b1 \quad \mathbf{S}(12,5) = -b2$$

$$\mathbf{S}(12,8) = -b3$$

The modulator's signal transfer function is easily obtained from (1) and looks as follows^

$$STF(z) = \frac{(-1)^{(p+q)}\Delta_{p,q}}{\det \mathbf{S}} = \frac{\sum_{i=1}^{N} m_i z^{-i}}{\sum_{j=0}^{N} n_j z^{-j}} \qquad (2)$$

$$NTF(z) = \frac{(-1)^{(r+q)}\Delta_{r,q}}{\det \mathbf{S}} = \frac{(1-z^{-1})^{N}}{\sum_{j=0}^{N} n_j z^{-j}} \qquad (3)$$

where N=4 – order of the modulator, p-signal input node number (in this graph 13), q-output node number (12), r – noise input node number (12).

The numerator and denominator coefficients are calculated from the demodulator schematic coefficients with the following formulae:

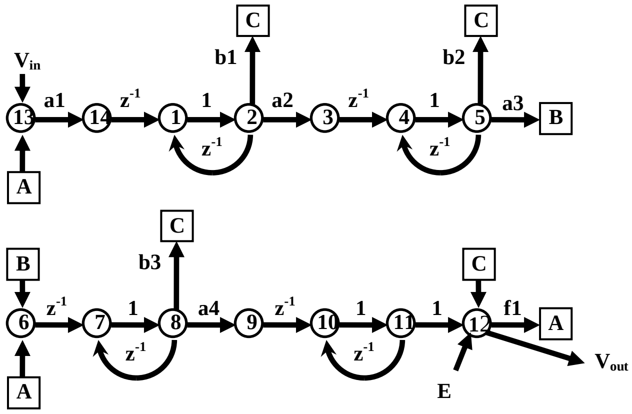

Fig. 1. Flow graph for the distributed feedforward topology.

$$m_4 = -(a1*b1 - a1*a2*b2 - a1*a2*a3*a4 + \\ + a1*a2*a3*b3)$$

$$m_3 = -(2*a1*a2*b2 - 3*a1*b1 - \\ - a1*a2*a3*b3)$$

$$m_2 = -(3*a1*b1 - a1*a2*b2)$$

$$m_1 = a1*b1$$

$$n_4 = a1*b1*f1 - a1*a2*b2*f1 - \\ - a1*a2*a3*a4*f1 + \\ + a1*a2*a3*b3*f1 + 1$$

$$n_3 = 2*a1*a2*b2*f1 - 3*a1*b1*f1 - \\ - a1*a2*a3*b3*f1 - 4$$

$$n_2 = 3*a1*b1*f1 - a1*a2*b2*f1 + 6$$

$$n_1 = - a1*b1*f1 - 4$$

$$n_0 = 1$$

It is easy to show that these formulas are the same as obtained in [2] by visual analysis of modulator structure.

Sigma-delta modulator analysis pays the main attention to the analysis of the noise magnitude response. It is this response that determines the achievable signal to noise ratio and therefore the effective conversion word length (effective number of bits, ENOB).

Another example is the 3rd order sigma-delta modulator designed for the high quality digital to analog conversion of audio signals [1].

Fig.2 shows the functional schematics of the modulator. To obtain it's transfer function and signal and noise magnitude responses we develop the flow graph shown in Fig.3.

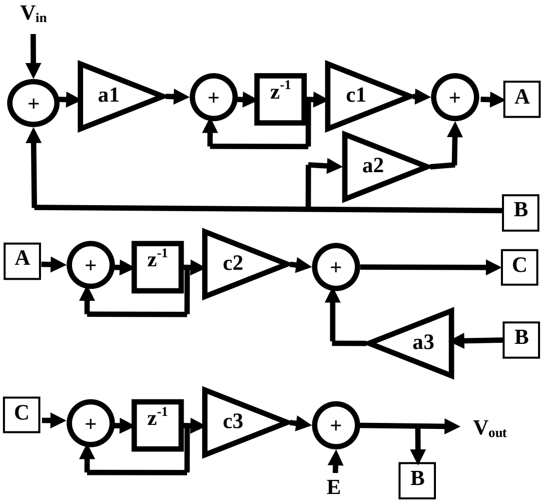

Fig. 2. 3rd order sigma-delta modulator schematic diagram.

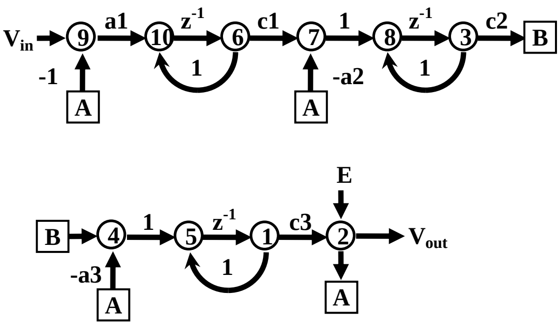

Fig. 3. 3rd order sigma-delta modulator flow graph.

Non zero elements of the system matrix are presented below.

$$\mathbf{S}(\mathbf{i},\mathbf{i}) = 1, i = 1 \ldots 10$$

$$\mathbf{S}(5,1) = \mathbf{S}(5,4) = \mathbf{S}(8,3) = \mathbf{S}(8,7) == \mathbf{S}(10,6) = -1$$

$$\mathbf{S}(9,2) = 1$$

$$\mathbf{S}(1,5) = \mathbf{S}(3,8) = \mathbf{S}(6,10) = -z^{-1}$$

$$\mathbf{S}(2,1) = -c3 \quad \mathbf{S}(4,2) \ = a3 \ \ \mathbf{S}(4,3) \ = -c2$$

$$\mathbf{S}(7,2) = a2 \quad \mathbf{S}(7,6) \ = -c1 \ \ \mathbf{S}(10,9) = -a1$$

Signal and noise transfer functions are derived from (2) and (3) with N=3, p=9, q=2, r=2.

The transfer functions coefficients are determined by the modulator coefficients as follows:

$$m_3 = \mathrm{a1} * \mathrm{c1} * \mathrm{c2} * \mathrm{c\,3}$$

$$m_2 = m_1 = 0$$

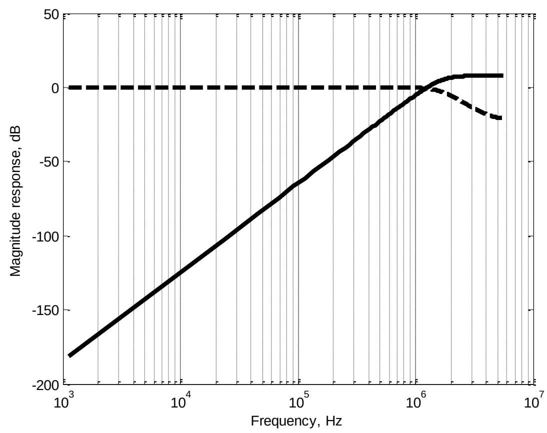

Fig. 4. STF (dashed line) and NTF(solid line) of the 3rd order modulator.

$$n_3 = -(\mathrm{a3} * \mathrm{c3} - \mathrm{a2} * \mathrm{c2} * \mathrm{c3} +$$

$$+ \mathrm{a1} * \mathrm{c1} * \mathrm{c2} * \mathrm{c3} + 1)$$

$$n_2 = -(\mathrm{a2} * \mathrm{c2} * \mathrm{c3} - 2 * \mathrm{a3} * \mathrm{c3} - 3)$$

$$n_1 = - \mathrm{a3} * \mathrm{c3} + 3$$

$$n_0 = 1$$

The values of modulator coefficients calculated in [1] are the following:

a1=0.0312;	c1=0.5;
a2=0.0617;	c2=1.0;
a3=0.0909;	c3=18.7539;

Signal and noise magnitude responses of the modulator are shown in Fig.4. The noise power level over the signal frequency band (0-20 kHz) calculated from the noise magnitude response is -140 dB

IV. Conclusion

The matrix model of a digital circuit of arbitrary topology allows to derive automatically the signal and noise transfer functions of sigma-delta modulators under consideration. In addition this approach can be used in modulator design procedures. The later possibility should be investigated in the future.

References

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